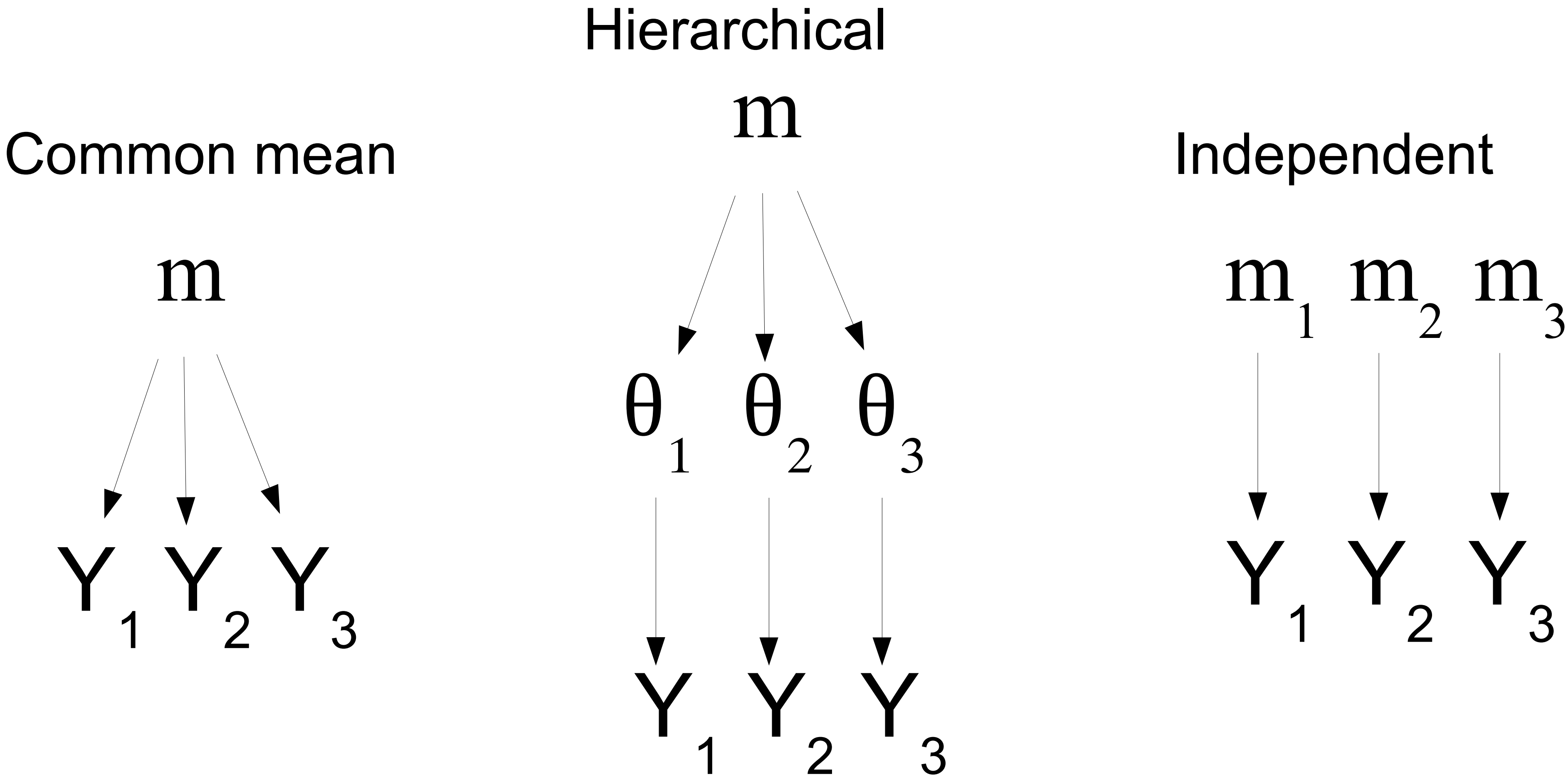
Hierarchical Bayes

Assumptions of Linear Model

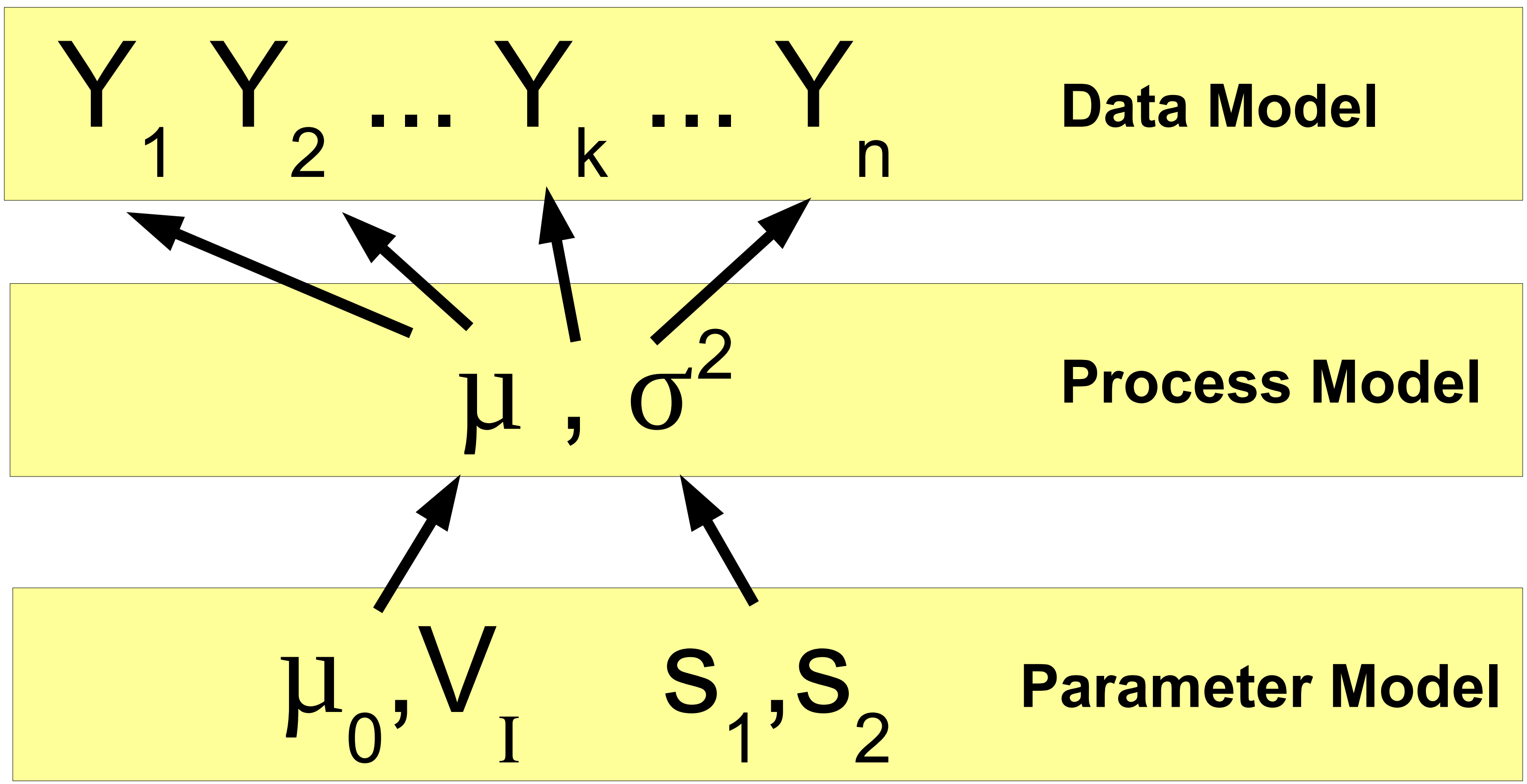
- Homoskedasticity **Model variance**
- No error in X variables **Errors in variables**
- No missing data **Missing data model**
- Normally distributed error **GLM**
- **Error in Y variables is measurement error**
- **Observations are independent**

Hierarchical Models



Common Mean

$$\vec{y}_k \sim N(\mu, \sigma^2)$$



Hierarchical Mean, Common Variance

$$Y_k \sim N(\mu_k, \sigma^2)$$

$$\mu_k \sim N(\mu, \tau^2)$$

$$\sigma^2 \sim IG(s_1, s_2)$$

At this point, this model is fitting each data set independently but assume the mean for each has the same prior

Hierarchical Mean, Common Variance

$$Y_k \sim N(\mu_k, \sigma^2)$$

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For the hierarchical model, instead assume the prior contains unknown model parameters

Hierarchical Mean, Common Variance

$$Y_k \sim N(\mu_k, \sigma^2)$$

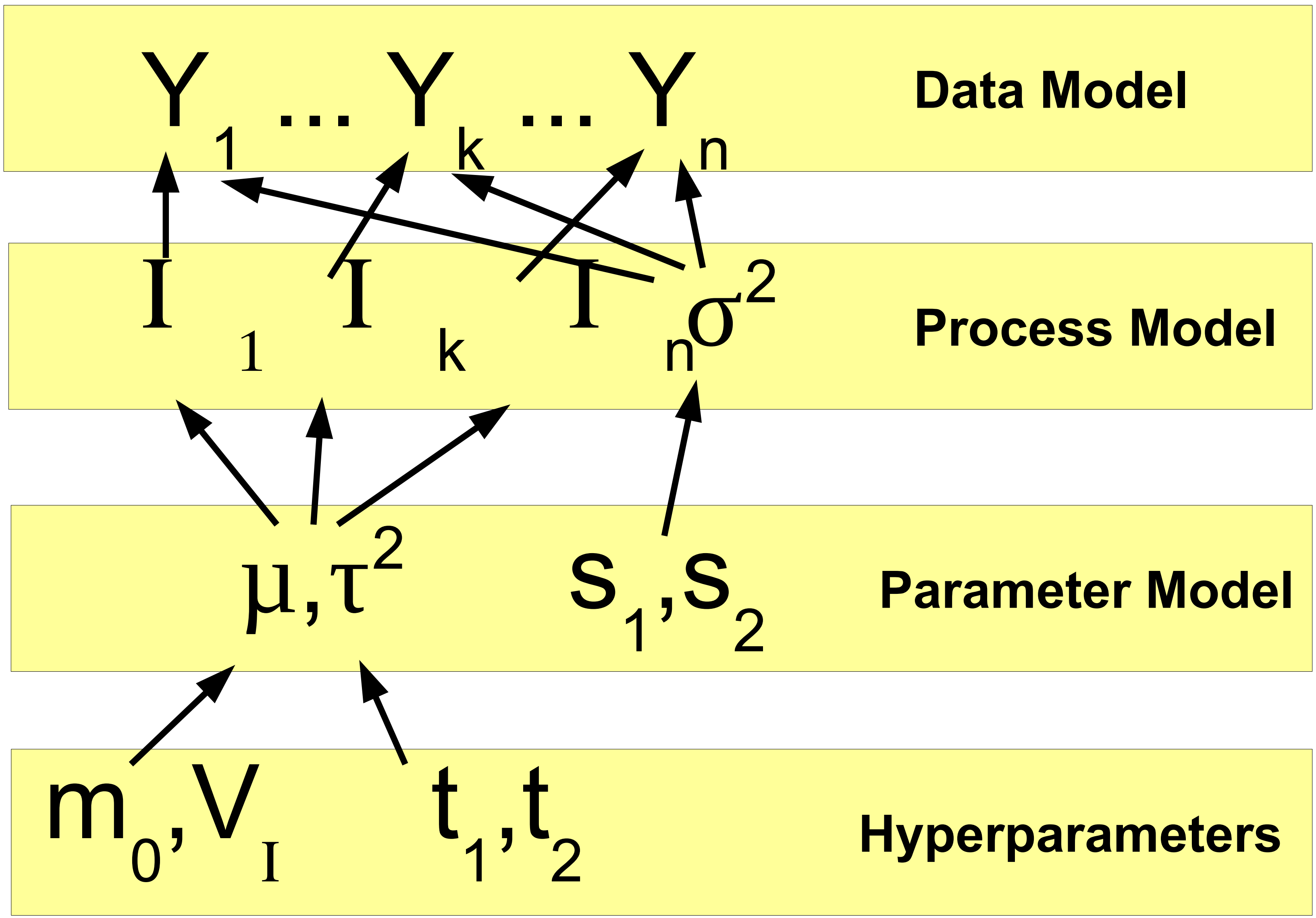
$$\mu_k \sim N(\mu, \tau^2)$$

$$\sigma^2 \sim IG(s_1, s_2)$$

$$\left.\begin{array}{l} \mu \sim N(\mu_{0,} V_\mu) \\ \tau^2 \sim IG(t_{1,} t_2) \end{array}\right\}$$

Then need to specify *hyperpriors* on our prior

Hierarchical Mean



Hierarchical Models

- Model variability in the parameters of a model
- Partition variability more explicitly into multiple terms
- Borrow strength across data sets

- Details usually in the SUBSCRIPTS
- Hierarchical with respect to parameters

Random Effects

- Common special case of Hierarchical models

$$Y_k \sim N(\mu_k, \sigma^2) \qquad Y_k \sim N(\mu_g + \alpha_k, \sigma^2)$$

$$\mu_k \sim N(\mu, \tau^2) \qquad \alpha_k \sim N(0, \tau^2)$$

$$\sigma^2 \sim IG(s_1, s_2) \qquad \sigma^2 \sim IG(s_1, s_2)$$

$$\mu \sim N(\mu_{0,} V_\mu) \qquad \mu_g \sim N(\mu_{0,} V_\mu)$$

$$\tau^2 \sim IG(t_{1,} t_2) \qquad \tau^2 \sim IG(t_{1,} t_2)$$

Random Effects

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Random Effects

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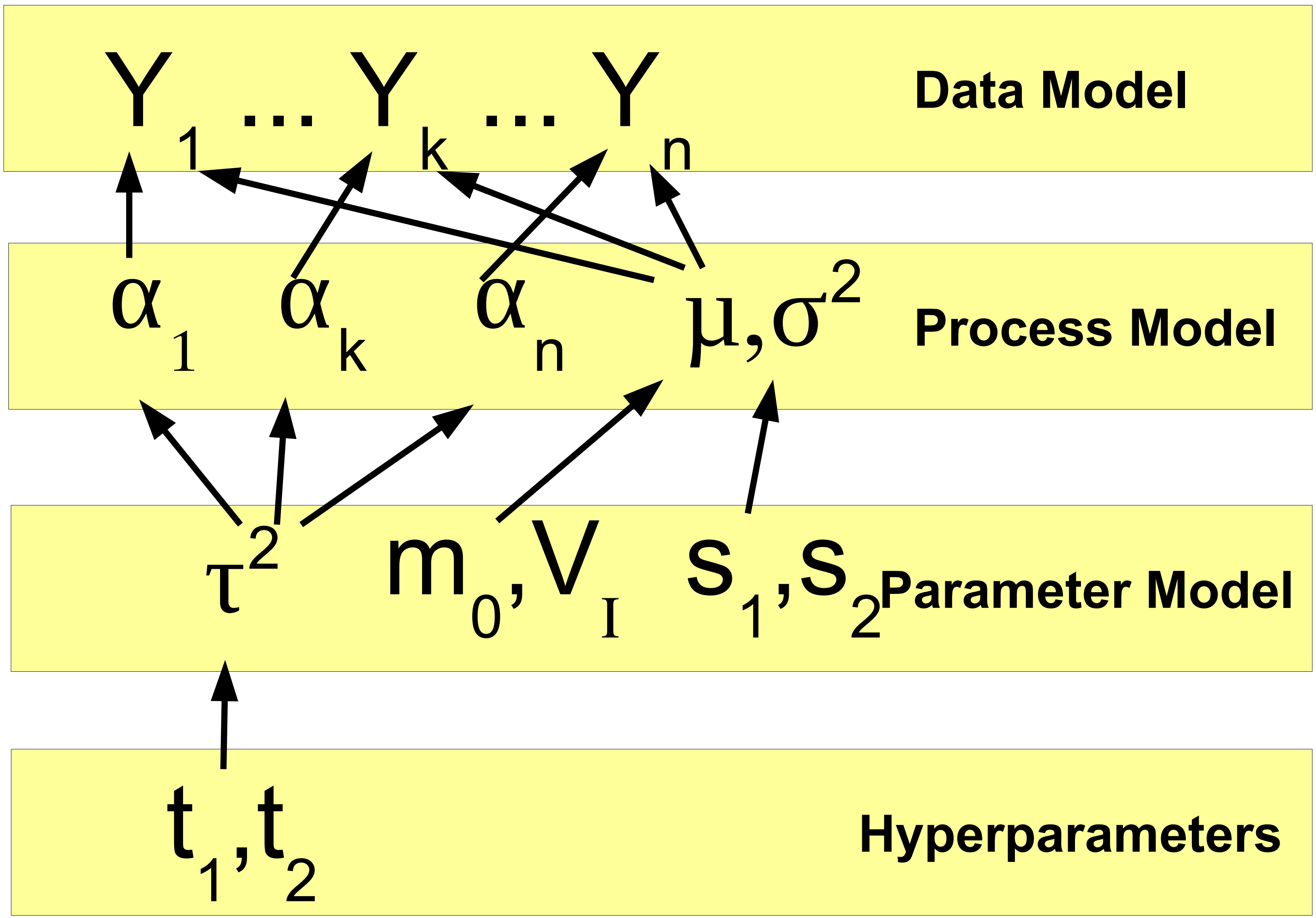
$$\tau^2 \sim IG(t_{1,} t_2)$$

- Random effects always have mean 0
- Random effects variance attributes a portion of uncertainty to a specific source
- Can be used to try an account for a lack of independence

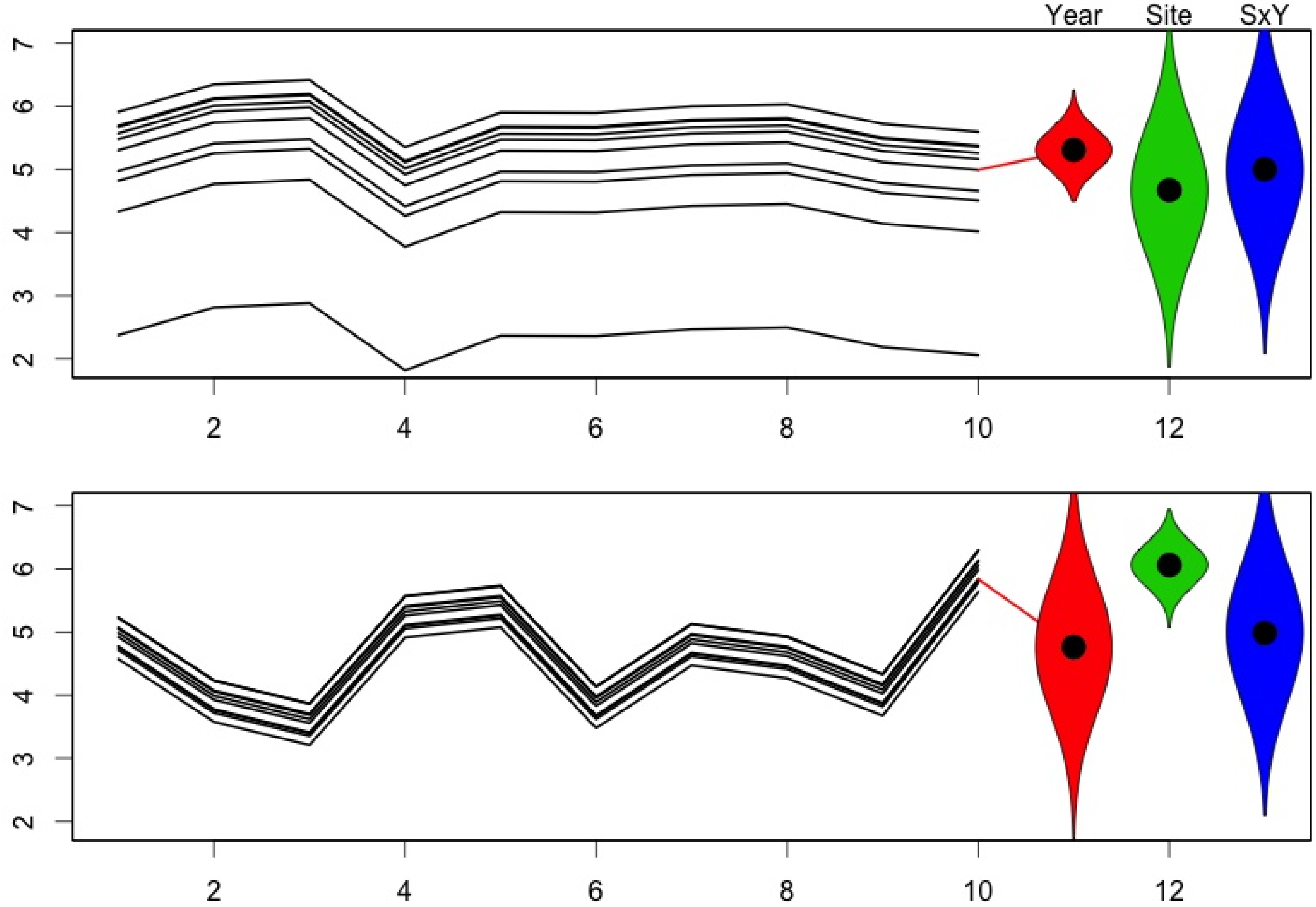
Random Effects Mean



What things can be random effects?

- Traditionally, random effects apply to aspects of the study that would not be the same if replicated
  - e.g. Plot, Block, Year, individual, etc.
  - Often used to account for a lack of independence
- Treatments and covariates of interest are usually treated as **fixed effects**
- Typically there is some degree of replication otherwise the random effect is not identifiably different from the residual “noise” term $J \sim N(0,\sigma^2)$

Why bother?
Impacts on inference...



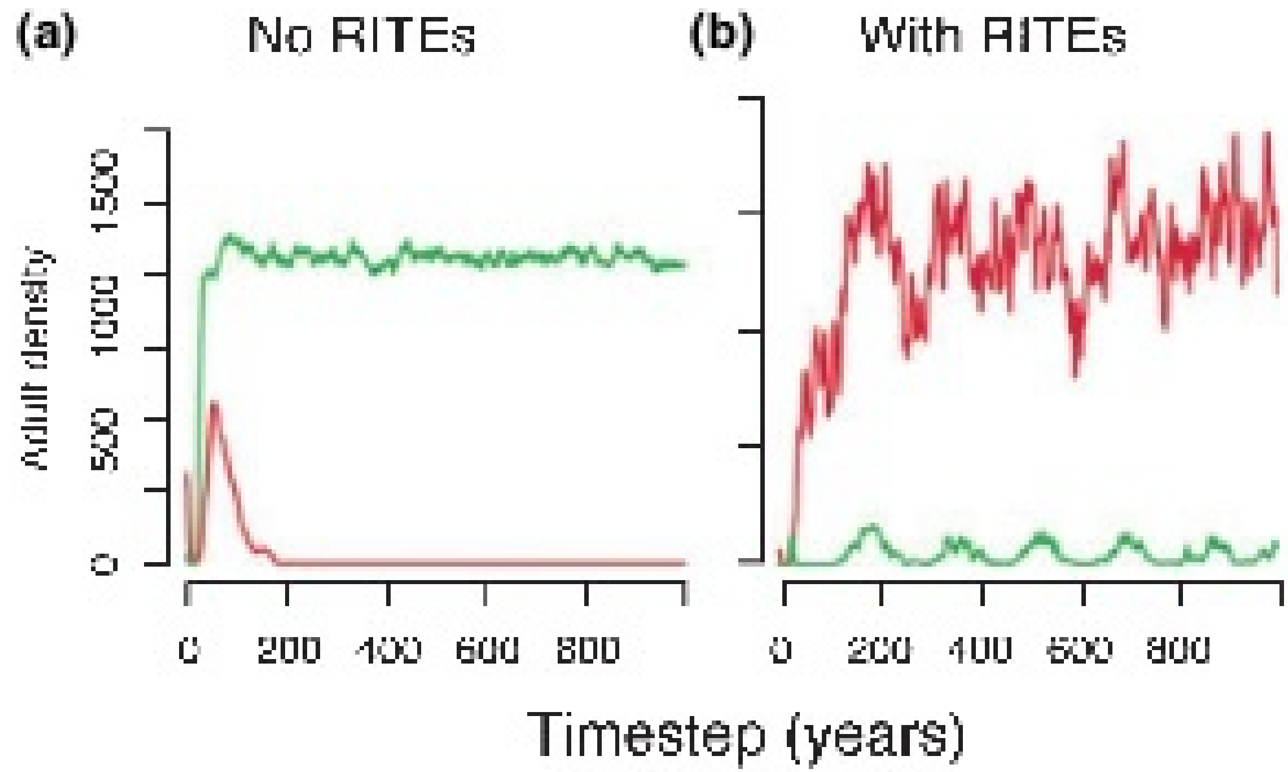
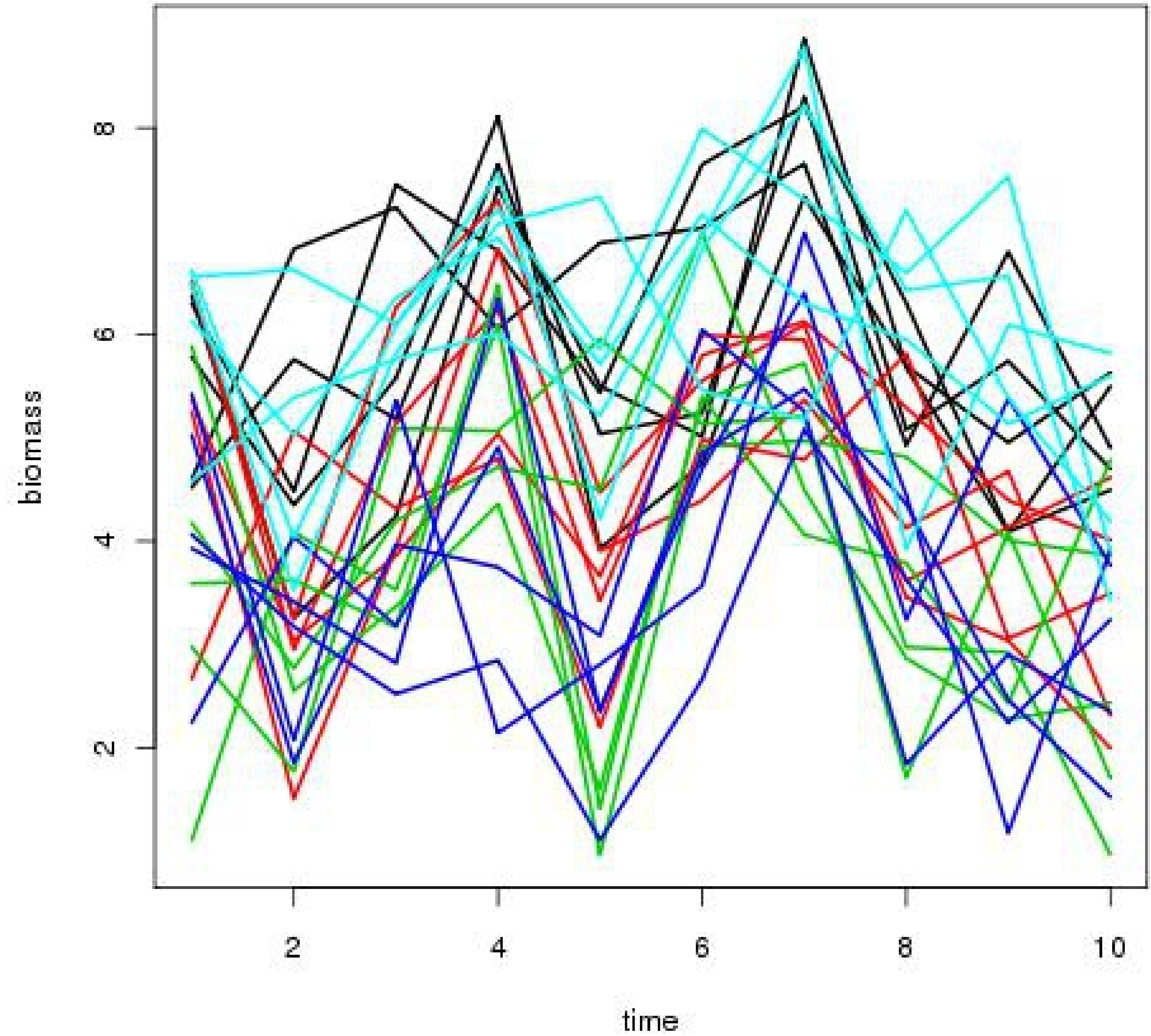


Figure 3 The impact of random individual effects (RITEs) on coexistence of two competing species. Two spatiotemporal and individual-based simulations were run using recruitment processes that are parameterized with data, summarized in Fig. 1. Panel (a) is the traditional approach having deterministic species differences and stochasticity in time, but no within-population heterogeneity, reflecting that fact the green species is the deterministic winner (Fig. 1a). Population heterogeneity in (b) means that green is not the deterministic winner, but rather both species win with some probability.

Start Simple

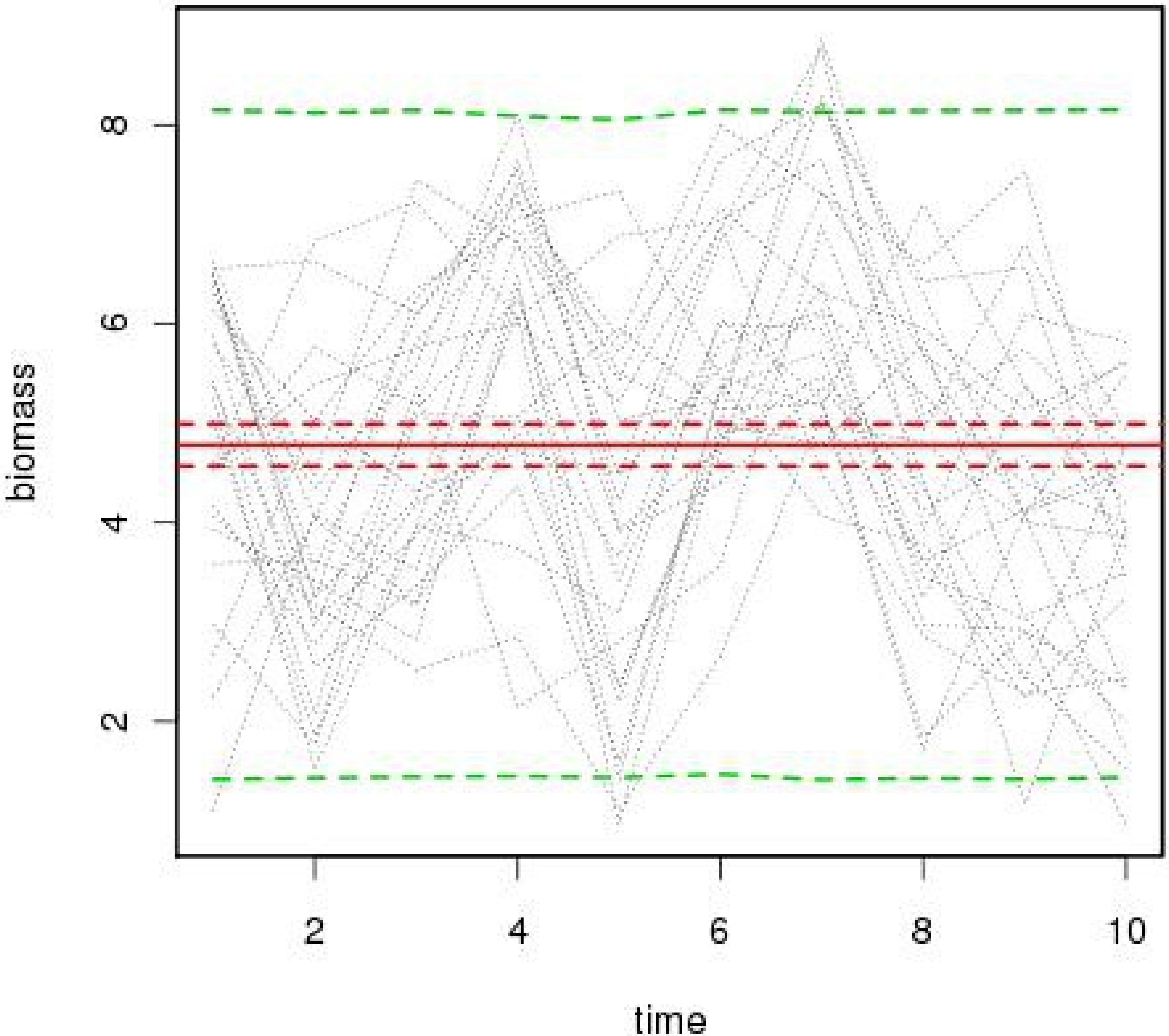
Progressively Add Complexity

Example: Biomass by Block and Time



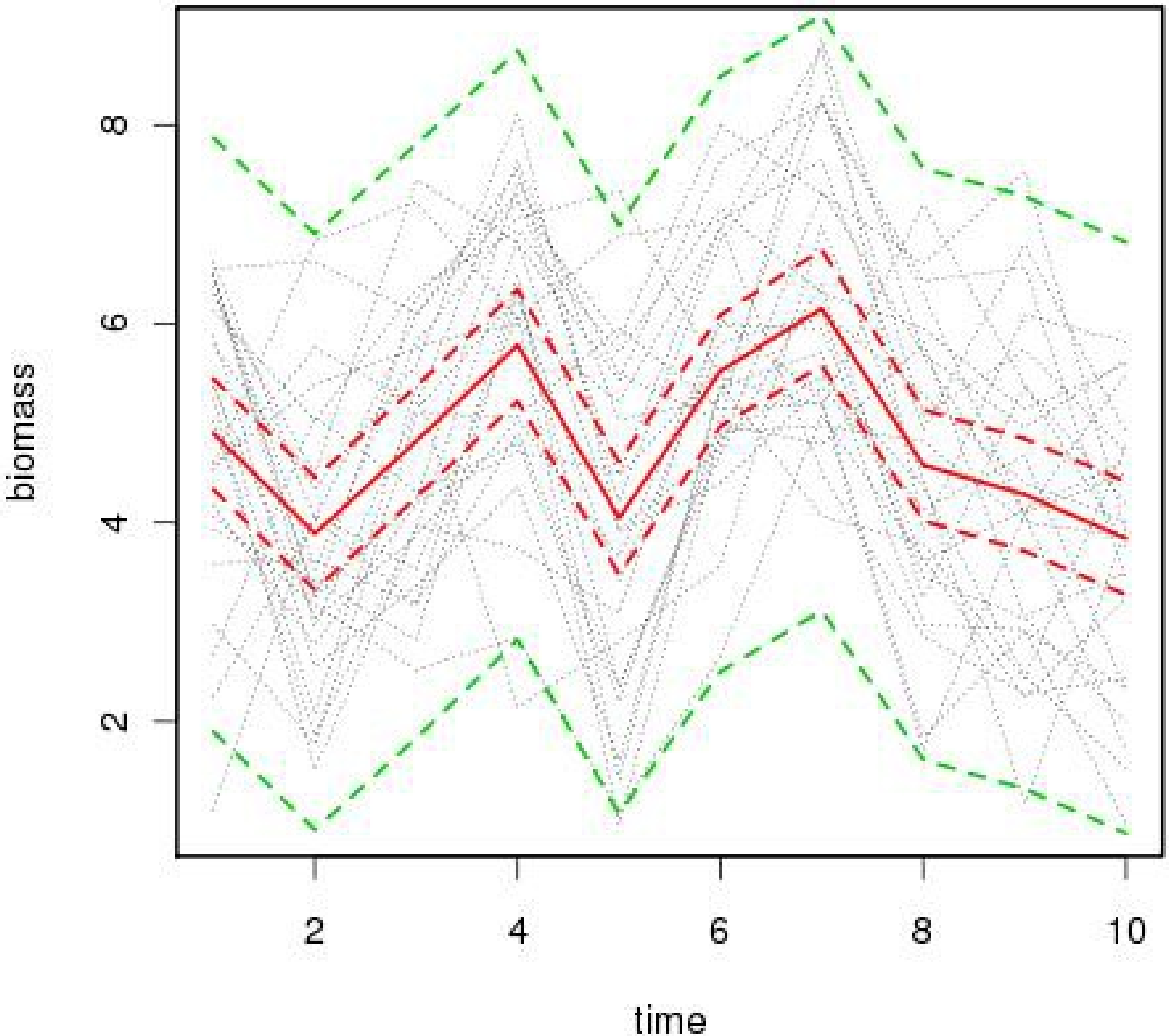
Model 1: Global Mean

```
model{
  mu ~ dnorm(0,0.001)                ## priors
  sigma ~ dgamma(0.001,0.001)

  for(t in 1:nt){           ## time
    for(b in 1:nb){         ## block
      for(i in 1:nrep){     ## individual
        x[t,b,i] ~ dnorm(mu,sigma)
      }
    }
   }
}
```


biomass
2
4
6
8
time
2
4
6
8
10

Model 2: Random Temporal Effect

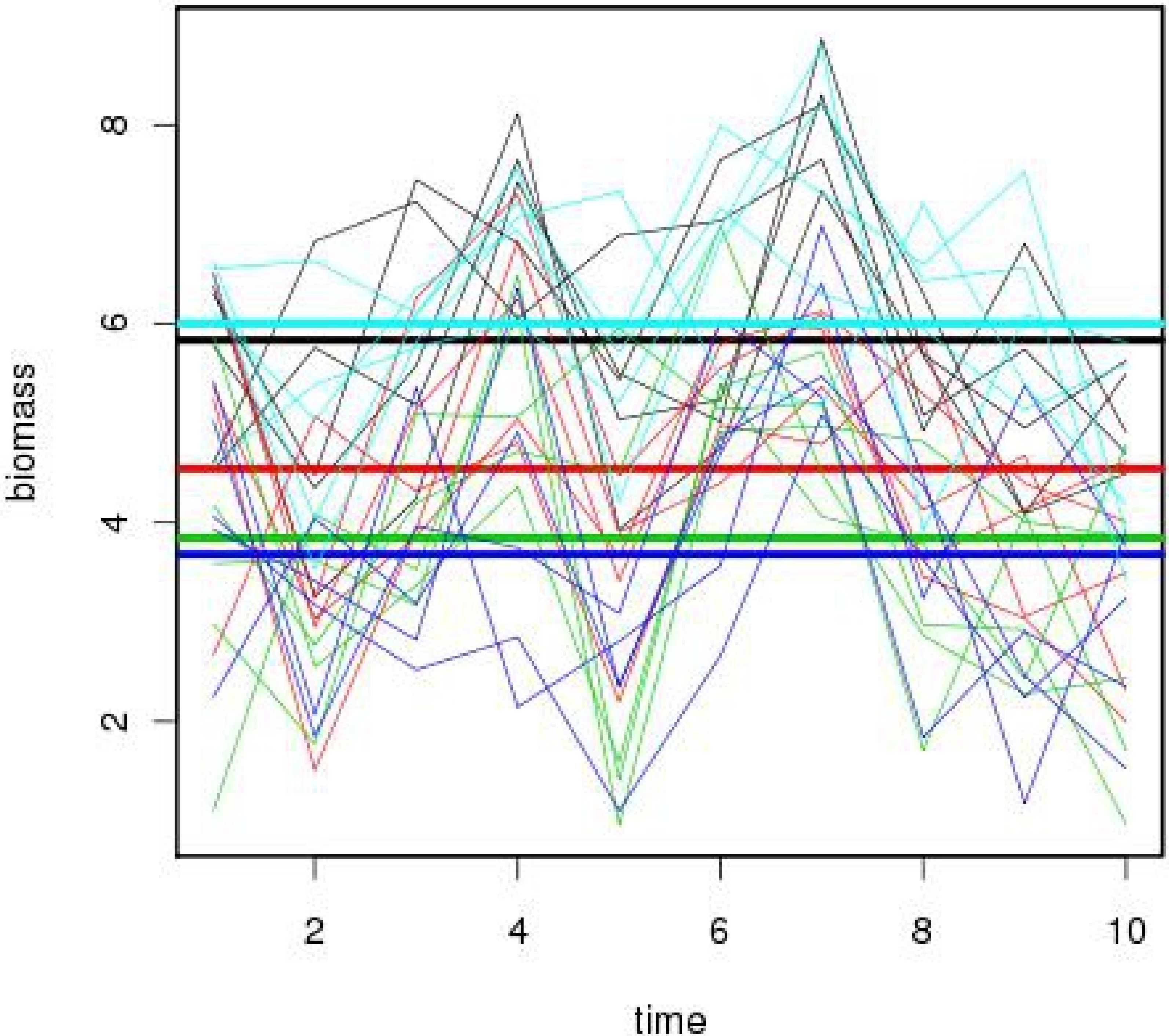
```
model{
  mu ~ dnorm(0,0.001)                    ## priors
  sigma ~ dgamma(0.001,0.001)
  for(t in 1:nt){alpha.t[t] ~ dnorm(0,tau.t)}
  tau.t ~ dgamma(0.001,0.001)            ## hyperprior

  for(t in 1:nt){
    Ex[t] <- mu + alpha.t[t]             ## process model
    for(b in 1:nb){
      for(i in 1:nrep){
        x[t,b,i] ~ dnorm(Ex[t],sigma)   ## data model
      }
    }
  }
}
```



biomass
time
2
4
6
8
2
4
6
8
10

Model 3: Random Block Effect

```
model{
  mu ~ dnorm(0,0.001)           ## priors
  sigma ~ dgamma(0.001,0.001)
  tau.b ~ dgamma(0.001,0.001)
  for(b in 1:nb){ alpha.b[b] ~ dnorm(0,tau.b)}

  for(b in 1:nb){
    Ex[b] <- mu + alpha.b[b]
    for(t in 1:nt){
      for(i in 1:nrep){
        x[t,b,i] ~ dnorm(Ex[b],sigma)
      }
    }
  }
}
```


biomass
2
4
6
8
time
2
4
6
8
10

Model 4: Random Block & Time

```
model{

  mu ~ dnorm(0,0.001)              ## priors
  sigma ~ dgamma(0.001,0.001)
  tau.b ~ dgamma(0.001,0.001)
  tau.t ~ dgamma(0.001,0.001)
  for(t in 1:nt){alpha.t[t] ~ dnorm(0,tau.t) }
  for(b in 1:nb){alpha.b[b] ~ dnorm(0,tau.b) }

  for(t in 1:nt){
    for(b in 1:nb){
      Ex[t,b] <- mu + alpha.b[b] + alpha.t[t]
      for(i in 1:nrep){
        x[t,b,i] ~ dnorm(Ex[t,b],sigma)
      }
    }
  }
}
```

Summary Table

Model	mu	sigma	tau.t	tau.b	DIC
Global Mean	4.78 (0.11)	2.92 (0.27)			977.9
Random Time	4.75 (0.33)	2.23 (0.21)	0.97 (0.64)		919.8
Random Block	4.82 (0.69)	1.92 (0.18)		2.36 (3.62)	878.0
Random B x T	4.85 (0.75)	0.84 (0.08)	1.31 (0.67)	0.80 (0.60)	766.8

Mixed Model

	Fixed Effects	Random Effect	Residual Error	
$$\mu_{i,k} = X_i \beta + \alpha_k + \epsilon_{i,k}$$				**Process model**

$$\mu_{i,k} = X_i \beta + \alpha_k + \epsilon_{i,k}$$ **Process model**

$$\epsilon_{i,k} \sim N(0, \sigma^2)$$ **Data model**

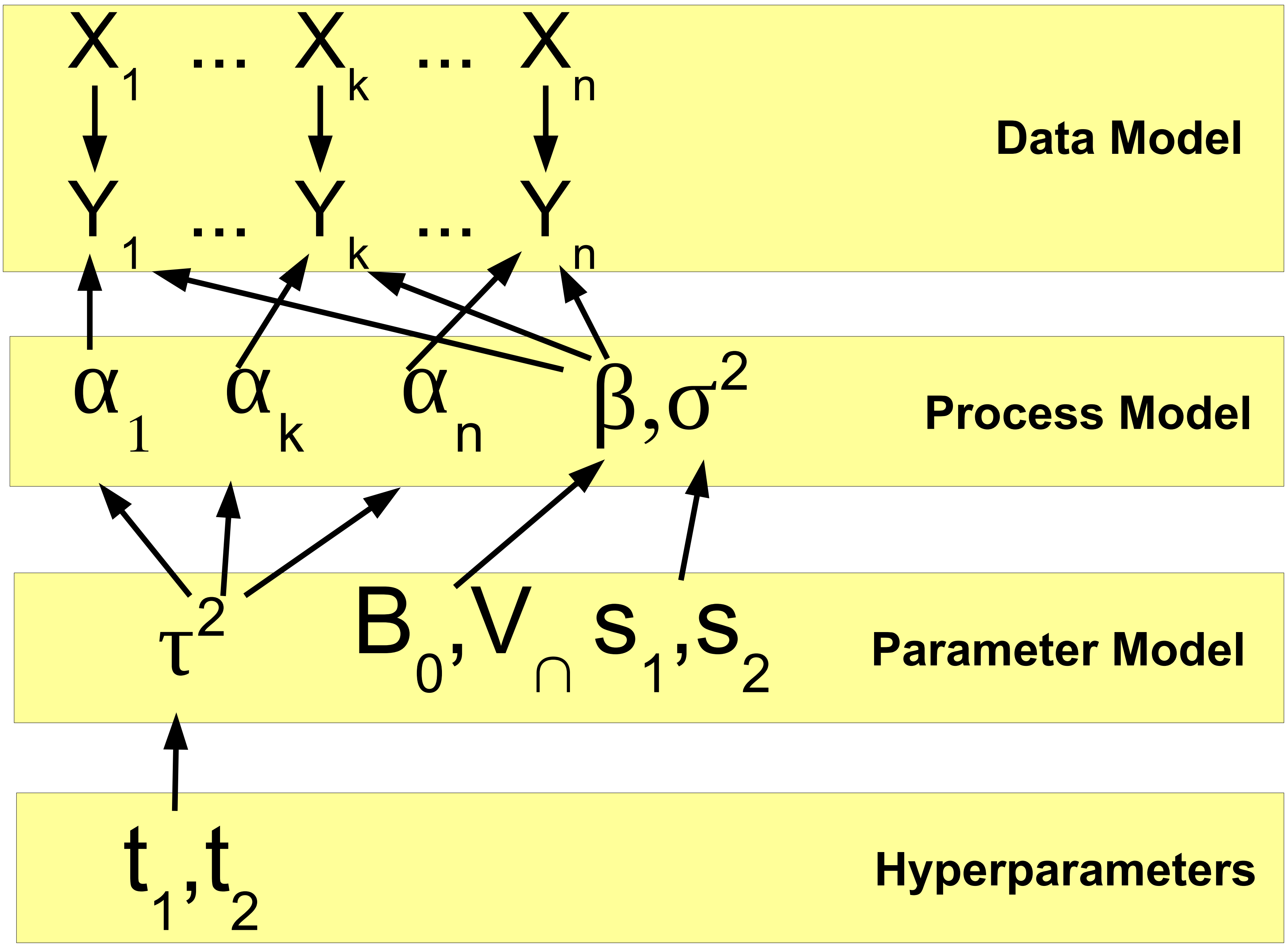
$$\alpha_k \sim N(0, \tau^2)$$ **Random effect**

$$\sigma^2 \sim IG(s_1, s_2)$$ **Error variance prior**

$$\beta \sim N(B_{0,} V_\beta)$$ **Fixed effects prior**

$$\tau^2 \sim IG(t_{1,} t_2)$$ **Random effects variance prior**

Mixed Model



X1 ... Xk ... Xn
Y1 ... Yk ... Yn
Data Model
α1 αk αn β,σ2
Process Model
τ2 B0,V∩ s1,s2
Parameter Model
t1,t2
Hyperparameters

Why bother?
Impacts on inference...

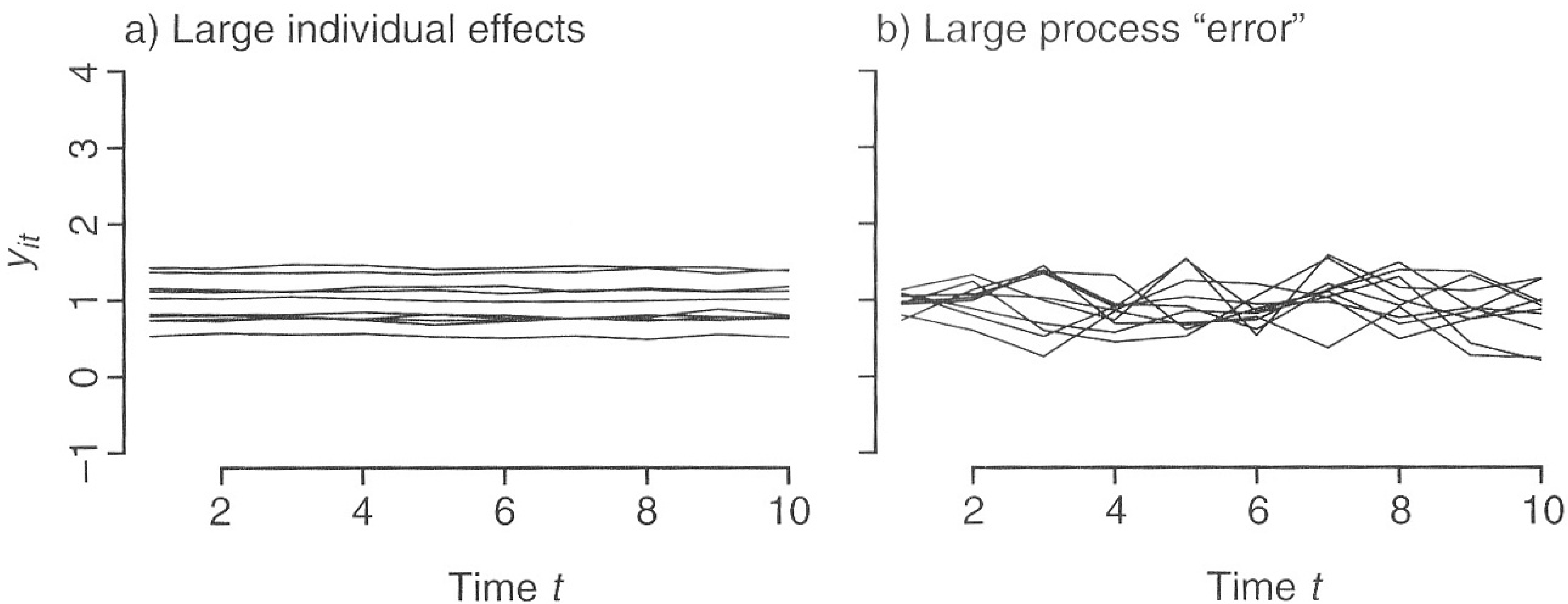


FIGURE 8.5. Two simulated longitudinal data sets with $n = 10$ and the same total variance, but dominated by individual differences (a) or process error (b). Variance parameters in (a) are $\tau^2 = 0.09$ and $\sigma^2 = 0.01$ and in (b) are $\tau^2 = 0.01$ and $\sigma^2 = 0.09$.

Explaining unexplained variance

- Random effects attempt to account for the unexplained variance associated with some group (plot, year, etc.) due to all the things that were not measured
- May point to scales that need additional explanation
- Adding covariates may explain some portion of this variance, but there's always something you didn't measure
- Sometimes additional fixed effects not justified (model selection)

Example: Year effects

- Consider the number of new young produced per adult female from population of birds
- Suppose adding a year effect shows significant year-to-year variability that is coherent through the whole population
- Based on the estimates of the year effects, could look for additional covariates that correlate with these values (e.g. different climate variables) without having to rerun the whole model
- Could refine the model to add additional drivers

Modeling Uncertainty

- Overall take home message:

  The proper accounting of uncertainty can be JUST AS IMPORTANT to making valid inference from your model as the process model and covariates

- Random effects are used to account for the impacts of unmeasured/unmeasurable covariates

Assumptions of Linear Model

- Homoskedasticity **Model variance**
- No error in X variables **Errors in variables**
- No missing data **Missing data model**
- Normally distributed error **GLM**
- Residual error in Y variables is measurement error
- Observations are independent

(last two:) **Hierarchical Models**